

ON THE ALMOST EVERYWHERE CONTINUITY

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ABSTRACT. The aim of this paper is to provide characterizations of the Lebesgue-almost everywhere continuity of a function $f : [a, b] \rightarrow \mathbb{R}$. These characterizations permit to obtain necessary and sufficient conditions for the Riemann integrability of f .

Key Words: continuity, function of one real variable, Riemann integrability.
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1. INTRODUCTION

The main aim of this paper is to establish the following theorem.

Theorem 1.1. *Let $a > b$ be two real numbers, and $f : [a, b] \rightarrow \mathbb{R}$ be a function. We assume that f admits a finite right-hand limit at each point of $[a, b)$ except on a Lebesgue-negligible set (respectively on a at most countable set). Then f is continuous at each point of $[a, b]$ except on a Lebesgue-negligible set (respectively on a at most countable set).*

The origine of this work is a paper of Daniel SAADA [4] which states that a real function defined on a real segment which right-hand continuous possesses at most an at most countable subset of discontinuity points. Saada attributes the proof of this result to Alain RÉMONDIÈRE. Studying this result and its proof, we see that it contains a central argument that we have described in our Lemma 4.1 and Lemma 4.2, and we use this argument to obtain other results. And so the present work is a continuation of the work of Rémondère and Saada.

In Section 2 we precise our notation and we give comments on them. In Section 3, we establish lemmas which are useful for the proof of Theorem 1.1. In Section 4, we provide results on the left-hand continuity and on the right-hand continuity. In Section 5 we give the proof of Theorem 1.1. In Section 6 we establish corollaries of Theorem 1.1.

2. NOTATION

We use the left-hand oscillation of f at $x \in (a, b]$ defined by

$$\omega_L(x) := \lim_{h \rightarrow 0+} \left(\sup_{y \in [x-h, x]} f(y) \right) - \lim_{h \rightarrow 0+} \left(\inf_{y \in [x-h, x]} f(y) \right)$$

and also the right-hand oscillation of f at $x \in [a, b)$ defined by

$$\omega_R(x) := \lim_{h \rightarrow 0+} \left(\sup_{y \in [x, x+h]} f(y) \right) - \lim_{h \rightarrow 0+} \left(\inf_{y \in [x, x+h]} f(y) \right).$$

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Note that we have $\sup_{y \in [x-h, x]} f(y) \geq f(x)$ and consequently

$$\lim_{h \rightarrow 0^+} \left(\sup_{y \in [x-h, x]} f(y) \right) \geq f(x) > -\infty.$$

Also note that we have $\inf_{y \in [x-h, x]} f(y) \leq f(x)$ and consequently

$$\lim_{h \rightarrow 0^+} \left(\inf_{y \in [x-h, x]} f(y) \right) \leq f(x) < +\infty.$$

And so, $\omega_L(x)$ is a sum of two elements of $(-\infty, +\infty]$ and therefore it is well-defined in $(-\infty, +\infty]$; more precisely its belongs to $[0, +\infty]$. For similar reasons, $\omega_R(x)$ is well-defined in $[0, +\infty]$.

We use the following notation when $g : [a, b] \rightarrow (-\infty, +\infty]$ and $r \in \mathbb{R}$:
 $\{g = 0\} := \{x \in [a, b] : g(x) = 0\}$, $\{g > r\} := \{x \in [a, b] : g(x) > r\}$,
 $\{g \leq r\} := \{x \in [a, b] : g(x) \leq r\}$.

A subset $N \subset [a, b]$ is called *Lebesgue-negligible* when there exists B , a borelian subset of $[a, b]$, such that $N \subset B$ and $\mu(B) = 0$ where μ denotes the Lebesgue measure of \mathbb{R} . Such a vocabulary is used for instance in [1].

Remark 2.1. *The following equivalence hold.*

- (A) When $x \in (a, b]$, $\omega_L(x) = 0$ if and only if f is left-hand continuous at x .
- (B) When $x \in [a, b)$, $\omega_R(x) = 0$ if and only if f is right-hand continuous at x .
- (C) When $x \in (a, b)$, $(\omega_L(x) = 0 \text{ and } \omega_R(x) = 0)$ if and only if f is continuous at x .

These equivalences are easy to prove. One important fact is that x belongs to the neighborhoods $[x - h, x]$ and $[x, x + h]$.

Remark 2.2. *When we will speak of the left-hand limit (respectively of the right-hand limit) of the function f at x , we speak of the limit of $f(y)$ when $y \rightarrow x, y > x$ (respectively $y \rightarrow x, y < x$); the point x is not included into his "neighborhoods". The situation is different in the definition of the oscillations ω_L and ω_R . We denote $f(x-) := \lim_{y \rightarrow x, y < x} f(y)$ and $f(x+) := \lim_{y \rightarrow x, y > x} f(y)$.*

3. PRELIMINARIES

We establish lemmas which are useful to the proof of Theorem 1.1.

Lemma 3.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a function, and $z \in [a, b)$. We assume that f admits a finite right-hand limit at z . Then we have :*

$$\forall \epsilon > 0, \exists \lambda(z, \epsilon) > 0, \forall x \in (z, z + \lambda(z, \epsilon)], \omega_L(x) \leq \epsilon.$$

Proof. We arbitrarily fix $\epsilon > 0$. Using the assumption, there exists $d_z \in \mathbb{R}$ such that

$$\exists \eta(z, \epsilon) > 0, \forall x \in [a, b], z < x \leq z + \eta(z, \epsilon) \implies |f(x) - d_z| \leq \epsilon. \quad (3.1)$$

When $x \in (z, z + \eta(z, \frac{\epsilon}{4}))$ and when $y \in (z, x]$, we have $y \in (z, z + \eta(z, \frac{\epsilon}{4}))$, and using (3.1) we obtain $|f(x) - d_z| \leq \frac{\epsilon}{4}$ and $|f(y) - d_z| \leq \frac{\epsilon}{4}$ that implies

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - d_z| + |f(y) - d_z| \leq 2\frac{\epsilon}{4} = \frac{\epsilon}{2} \implies \\ f(x) - \frac{\epsilon}{2} &\leq f(y) \leq f(x) + \frac{\epsilon}{2}. \end{aligned}$$

Then, for all $h \in (0, x - z]$, we obtain

$$\begin{aligned} f(x) - \frac{\epsilon}{2} &\leq \inf_{y \in [x-h, x]} f(y) \leq \sup_{y \in [x-h, x]} f(y) \leq f(x) + \frac{\epsilon}{2} \implies \\ f(x) - \frac{\epsilon}{2} &\leq \lim_{h \rightarrow 0+} \left(\inf_{y \in [x-h, x]} f(y) \right) \leq \lim_{h \rightarrow 0+} \left(\sup_{y \in [x-h, x]} f(y) \right) \leq f(x) + \frac{\epsilon}{2} \\ &\implies 0 \leq \omega_L(x) \leq f(x) + \frac{\epsilon}{2} - (f(x) - \frac{\epsilon}{2}) = \epsilon. \end{aligned}$$

And so it suffices to take $\lambda(z, \epsilon) := \eta(z, \frac{\epsilon}{4})$. \square

Using similar arguments we can prove the following result.

Lemma 3.2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a function, and $z \in (a, b]$. We assume that f admits a finite left-hand limit at z . Then we have :*

$$\forall \epsilon > 0, \exists \nu(z, \epsilon) > 0, \forall x \in [z - \nu(z, \epsilon), z), \omega_R(x) \leq \epsilon.$$

Lemma 3.3. *Let I be a nonempty set, and $(S_i)_{i \in I}$ be a family of subintervals of $[a, b]$ such that $S_i \cap S_j = \emptyset$ when $i \neq j$, and such $\mu(S_i) > 0$ for all $i \in I$, where μ denotes the Lebesgue measure of \mathbb{R} . Then I is at most countable.*

Proof. Since a positive measure is additive, for all finite subset $J \subset I$, we have $\mu(\cup_{j \in J} S_j) = \sum_{j \in J} \mu(S_j)$. Since a positive measure is monotonic, $\cup_{j \in J} S_j \subset [a, b]$ implies $\mu(\cup_{j \in J} S_j) \leq \mu([a, b]) = b - a$, and so we have $\sum_{j \in J} \mu(S_j) \leq b - a < +\infty$ for all finite subset J of I . Therefore the family of non negative real numbers $(\mu(S_i))_{i \in I}$ is summable in $[0, +\infty)$, and consequently the set $\{i \in I : \mu(S_i) \neq 0\}$ is at most countable (Corrolary 9-9, p. 220 in [2]). Since $\mu(S_i) > 0$ for all $i \in I$, we obtain that I is at most countable. \square

Remark 3.4. *We can also prove Lemma 3.3 by building a function $\varphi : I \rightarrow \mathbb{Q}$ in the following way: since \mathbb{Q} is dense into \mathbb{R} , for each $i \in I$, there exists $\varphi(i) \in \mathbb{Q} \cap S_i$. Since $S_i \cap S_j = \emptyset$ when $i \neq j$, we have $\varphi(i) \neq \varphi(j)$ when $i \neq j$. And so φ is injective. Since \mathbb{Q} is countable, $\varphi(I) \subset \mathbb{Q}$ is at most countable, and using an abridgement of φ , we build a bijection between $\varphi(I)$ and I .*

4. LIMITS ON ONE SIDE, CONTINUITIES ON THE OTHER SIDE

The following results establish that the existence of left-hand (respectively right-hand) limits implies the right-hand (respectively left-hand) continuity.

Lemma 4.1. *let $f : [a, b] \rightarrow \mathbb{R}$ be a function, and N be a Lebesgue-negligible (respectively at most countable) subset of $[a, b]$. We assume that f admits a finite right-hand limit at each $x \in [a, b] \setminus N$. Then the set of the points of $[a, b]$ where f is not left-hand continuous is Lebesgue-negligible (respectively at most countable).*

Proof. We arbitrarily fix $\epsilon > 0$. Using Lemma 3.1, denoting $\lambda_z := \lambda(z, \epsilon)$, we obtain the following assertion.

$$\forall z \in \{\omega_L > \epsilon\} \cap ([a, b] \setminus N), \exists \lambda_z > 0, (z, z + \lambda_z] \subset \{\omega_L \leq \epsilon\}. \quad (4.1)$$

Let $z_1, z_2 \in \{\omega_L > \epsilon\} \cap ([a, b] \setminus N)$, $z_1 \neq z_2$. We can assume that $z_1 < z_2$. After (4.1), we cannot have z_2 into $(z_1, z_1 + \lambda_{z_1}]$, therefore we have $z_2 > z_1 + \lambda_{z_1}$, and we have proven:

$$\forall z_1, z_2 \in \{\omega_L > \epsilon\} \cap ([a, b] \setminus N), z_1 \neq z_2 \implies (z_1, z_1 + \lambda_{z_1}] \cap (z_2, z_2 + \lambda_{z_2}] = \emptyset.$$

We have also $\mu((z, z + \lambda_z]) = \lambda_z > 0$. Then using Lemma 3.3, we can assert that

$$\forall \epsilon > 0, \{\omega_L > \epsilon\} \cap ([a, b] \setminus N) \text{ is at most countable.} \quad (4.2)$$

Note that $\{\omega_L > 0\} = \omega_L^{-1}((0, +\infty]) = \omega_L^{-1}(\bigcup_{n \in \mathbb{N}_*} (\frac{1}{n}, +\infty])$
 $= \bigcup_{n \in \mathbb{N}_*} \omega_L^{-1}((\frac{1}{n}, +\infty]) = \bigcup_{n \in \mathbb{N}_*} \{\omega_L > \frac{1}{n}\} \implies$

$$\{\omega_L > 0\} \cap ([a, b] \setminus N) = \bigcup_{n \in \mathbb{N}_*} (\{\omega_L > \frac{1}{n}\} \cap ([a, b] \setminus N)).$$

Using (4.2), since a countable union of at most countable subsets is at most countable, we obtain the following assertion.

$$\{\omega_L > 0\} \cap ([a, b] \setminus N) \text{ is at most countable.} \quad (4.3)$$

Note that $\{\omega_L > 0\} = (\{\omega_L > 0\} \cap ([a, b] \setminus N)) \cup (\{\omega_L > 0\} \cap N)$.

Since $(\{\omega_L > 0\} \cap N) \subset N$ and since N is Lebesgue-negligible (respectively at most countable), $(\{\omega_L > 0\} \cap N)$ is Lebesgue-negligible (respectively at most countable). Recall that an at most countable subset of \mathbb{R} is Lebesgue-negligible. And so when N is Lebesgue-negligible, $\{\omega_L > 0\}$ is Lebesgue-negligible as a union of two Lebesgue-negligible subsets, and when N is at most countable, $\{\omega_L > 0\}$ is at most countable as a union of two at most countable subsets. Using (A) of Remark 2.1, the lemma is proven. \square

Proceedings as in the proof of Lemma 4.1, we obtain the following result.

Lemma 4.2. *let $f : [a, b] \rightarrow \mathbb{R}$ be a function, and M be a Lebesgue-negligible (respectively at most countable) subset of $[a, b]$. We assume that f admits a finite left-hand limit at each $x \in (a, b] \setminus M$. Then the set of the points of $[a, b]$ where f is not right-hand continuous is Lebesgue-negligible (respectively at most countable).*

5. PROOF OF THEOREM 1.1

Using Lemma 4.1 and (A) of Remark 2.1, $\{\omega_L > 0\}$ is Lebesgue-negligible (respectively at most countable) since $\{\omega_L > 0\}$ is exactly the set of the points of $(a, b]$ where f is not left-hand continuous.

Now, setting $M = \{\omega_L > 0\}$, for all $x \in [a, b] \setminus M$, $f(x-) = f(x) \in \mathbb{R}$, and the assumption of Lemma 4.2 is fulfilled. Consequently we obtain that $\{\omega_R > 0\}$ is Lebesgue-negligible (respectively at most countable) after (B) of Remark 2.1.

Note that $\{\omega_L = 0\} \cap \{\omega_R = 0\}$ is exactly the set of the points of (a, b) where f is continuous. We have

$$[a, b] \setminus ((\{\omega_L = 0\} \cap \{\omega_R = 0\})) = [a, b] \cap ((\{\omega_L > 0\} \cup \{\omega_R > 0\})) = \{\omega_L > 0\} \cup \{\omega_R > 0\}.$$

This set is Lebesgue-negligible (respectively at most countable) as a union of two Lebesgue-negligible (respectively at most countable) sets. Note that $\{a, b\}$ is Lebesgue-negligible (respectively at most countable) and so set of the discontinuity points of f is Lebesgue-negligible (respectively at most countable).

6. CONSEQUENCES

A first consequence of Theorem 1.1 is the following result.

Theorem 6.1. *Let $a > b$ be two real numbers, and $f : [a, b] \rightarrow \mathbb{R}$ be a function. Then the following assertions are equivalent.*

- (α) *The set of the discontinuity points of f is Lebesgue-negligible (respectively at most countable).*
- (β) *The set of the left-hand discontinuity points of f is Lebesgue-negligible (respectively at most countable).*
- (γ) *The set of the right-hand discontinuity points of f is Lebesgue-negligible (respectively at most countable).*
- (δ) *The set of the points where f does not admit a finite left-hand limit is Lebesgue-negligible (respectively at most countable).*
- (ϵ) *The set of the points where f does not admit a finite right-hand limit is Lebesgue-negligible (respectively at most countable).*

Proof. The implications $(\alpha) \implies (\beta) \implies (\delta)$ are easy, and $(\delta) \implies (\alpha)$ is Theorem 1.1. The implications $(\alpha) \implies (\gamma) \implies (\epsilon)$ are easy. we can do a proof which is similar to this one of Theorem 1.1 to prove $(\epsilon) \implies (\alpha)$. \square

About the Riemann-integrability we recall a famous theorem of Lebesgue, [3] p. 29, [5] p. 20.

Theorem 6.2. *Let $a > b$ be two real numbers, and let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Then the following assertions are equivalent.*

- (i) *f is Riemann integrable on $[a, b]$.*
- (ii) *The set of the discontinuity points of f is Lebesgue-negligible.*

As a consequence of Theorem 6.1 and of the previous classical theorem of Lebesgue, we obtain the following result on the Riemann-integrability.

Theorem 6.3. *Let $a > b$ be two real numbers, and let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Then the following assertions are equivalent.*

- (a) *f is Riemann integrable on $[a, b]$.*
- (b) *The set of the points where f does not admit a finite left-hand limit is Lebesgue-negligible.*
- (c) *The set of the points where f does not admit a finite right-hand limit is Lebesgue-negligible.*

An easy consequence of this result is the following one.

Corollary 6.4. *Let $a > b$ be two real numbers, and let $f : [a, b] \rightarrow \mathbb{R}$ be a function. If f is right-hand continuous on $[a, b]$ or left-hand continuous on $[a, b]$, then the set of the discontinuity points of f is at most countable, and consequently when in addition f is assumed to be bounded, f is Riemann integrable on $[a, b]$.*

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